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# Two Algorithms Based on Shooting Method for Solution of Falkner-Skan Equation

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**Abstract**— Two algorithms based on shooting method have been developed, one for the solution of the Falkner-Skan [1] equation representing boundary layer flow past a wedge of angle β × π and the other for estimating the parameter β occurring in the governing equation. The numerical results agree upto 3 decimal places with the results obtained by previous authors using other numerical methods for the particular values β=0,1/2,1 and numerical results for β=-0.1988 are presented in table.

**Index Terms**— Algorithm. Shooting Method, Falkner-Skan Equation, boundary layer, similarlty variable, initial value problem, boundary value problem, viscous fluid.

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## 1 Introduction

The following equation known as Falkner-Skan [1] equation

$$f''' + f f'' + \beta (1 - f'^2) = 0 \qquad (1)$$

Under the boundary condition

$$f(0) = 0 , f'(0) = 0 , f'(\infty) = 1 \qquad (2)$$

represents the flow of viscous fluid past a wedge of angleβ. This equation was derived by Falkner-Skan [1] using dimensionless similarity variables in the boundary layer equations describing flow past a wedge. The equation is third order quasi linear and two conditions are specified at argument = 0 while one condition at argument = 1.

In this paper we present two algorithms one for the solution of Falkner-Skan [1] equation and the other for estimating the parameter $\beta$ and finally a numerical solution based on shooting method is presented.

## 2 Shooting Method

The Shooting method [2] is an advanced sophisticated computer oriented numerical method for solving boundary value problems. Let us consider a general two point third order boundary value problem. In all third order boundary value problem two boundary conditions are prescribed in one end point and another boundary condition at the other point. Without loss of generality let us assume that two boundary conditions are prescribed at the initial point $x = x_0$ and one boundary condition at the final point $x=x_f$ (the problem prescribed with two boundary conditions at the final point and one condition at the initial point can be treated similarly by backward interpolation.

Let the general third order two-point boundary value problem be

$$y''' = f(x, y, y', y'') \qquad (3)$$

With boundary conditions $y^{(p)}(x_0) = y_0^{(p)}$, p is any two of 0,1,2
And

$$y^{(p)}(x_f) = y_f^{(p)} , \text{ p is any one } 0,1,2 \qquad (4)$$

where the upper index p denotes the order of differentiation with respect to x and the lower index 0 for initial point and f for final point. That is we are considering the problem with two boundary conditions at the initial point and one boundary condition at the final point. We use here the initial value method for solving ordinary differential equation. But in order to do so we must know all the initial conditions needed. Since one of $y_0^{(p)}$ $^{(p=0,1,2)}$ is missing at $x=x_0,$ let it be an unknown parameter λ(say) which must be so determined that the resulting solution yields the prescribed final value $y_f^{(p)}$ $^{(p = 0, 1, 2)}$ at $x=x_f$ to some desired accuracy.

Let $\lambda_0$ and $\lambda_1$ be guess values of the missing initial condition $y_0^{(p)}(x_0)$ ( p takes the value 0, 1,2 which is not in the given initial condition). Let $y^{(P)}(x_f,\lambda_0)$ and $y^{(P)}(x_f,\lambda_1)$ be the values of $y^{(p)}(x_f)$ ( p is given to be any one of 0,1,2) for $\lambda=\lambda_0$ and $\lambda=\lambda_1$ respectively obtained on integration of the initial value problem for (3) in which $\lambda_k$ (k=0,1)is taken as the missing initial condition. Then geometrically a better approximation $\lambda_2$ of λ can be obtained as follows.

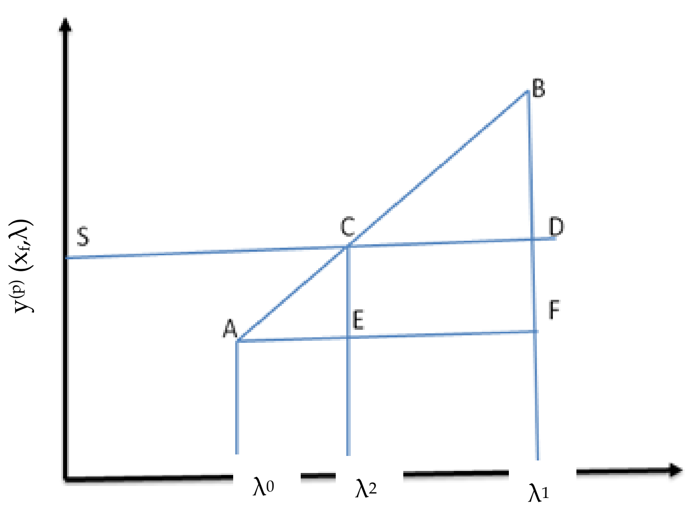


**Fig 1: Linear interpolation of Guess Values**

Let the points A and B in the Fig.1 represents the value of $y^{(p)}(x_f)$ when $\lambda = \lambda_0$ and $\lambda = \lambda_1$ respectively, when $y^{(p)}(x_f)$ is plotted against $\lambda$. Let SD be the line $y^{(p)}(x_f) = y^{(p)}{}_f$
then a better approximation of $\lambda_2$ of $\lambda$ can be obtained by linear interpolation given by

$$\lambda_2 = \lambda_0 + (\lambda_1 - \lambda_0) \{ y^{(p)}{}_f - y^{(p)} ( x_f , \lambda_0 )\} / \{ y^{(p)} ( x_f , \lambda_1 ) - y^{(p)} ( x_f , \lambda_0 )\} \quad (5)$$

(from the similar triangles CAE and BAF). Now (3) can be integrated with the two given initial conditions and $\lambda_2$ as the missing initial condition to obtain $y^{(p)} ( x_f , \lambda_2 )$. Again using linear interpolation based on $\lambda_2$ and $\lambda_1$ we can obtain the next approximation $\lambda_3$. The process is repeated until $y^{(p)} ( x_f , \lambda_k ) = y^{(p)}{}_f$ is satisfied to some desired accuracy for some k.

Convergence of the iteration process described above is not guaranteed. But once the chosen value is nearer to the true value the convergence is very rapid.

The shooting method has tremendous applications in solving boundary value problems. It is also applied to three- point double parametric boundary value problems by various authors. We have developed a general algorithm to solve (3) to (4) which which we have described in the next section.

## 3 Numerical Solutions

Due to nonlinearity of the problem (3) to (4) a closed form solution is not possible to obtain. Hence numerical solutions of the problem are essential. Based on the shooting method of a general algorithm has been developed to solve (3) to (4).

### 3.1 Algorithm

Step1 : Let $\lambda_k$ be the approximation for the unknown initial condition $y^{(p)} ( x_0 ) = y_0{}^{(p)}$ , p is any one of 0,1,2 and takes the value which is not in the other prescribed conditions. ($\lambda_0$ , $\lambda_1$ can be chosen from physical consideration of the problem or by a trial and error method).

Step2: Solve the initial value problem
$Y''' = f (x, y, y', y'')$
$y^{(p)} ( x_0 )= y_0{}^{(p)}$ , p is any two of 0,1,2 which are prescribed and $y^{(p)} (x_0)= \lambda_k$ , k= 0,1 .... and p is any one of 0,1,2 which are not in the prescribed conditions from $x=x_0$ to $x=x_f$ using any method for solving initial value problems.

Step3: Call $y^{(p)} (x_f, \lambda_k)$ , (k=0,1,.....) the value of $y^p (x_f)$ ( p is any one of 0,1,2 which is given ) obtain on integration the initial value problem in Step – 2.

Step 4: If $\left| y^{(p)}{}_f - y^{(p)} (x_f, \lambda_k) \right| < \varepsilon$ for a tolerable $\varepsilon$, then go to step 6 otherwise continue.

Step 5: Obtain the next approximation for the unknown initial condition from
$\lambda_{k+2} = \lambda_k + (\lambda_{k+1} - \lambda_k) \{ y^{(p)}{}_f - y^{(p)} ( x_f , \lambda_k )\} / \{ y^{(p)} ( x_f , \lambda_{k+1} ) - y^{(p)} ( x_f , \lambda_k )\}$ and go to step 2.

Step 6: Solve the problem using two given initial condition and $\lambda_k$ is the missing initial condition and stop. (though the convergence is not guaranteed at present in the iteration process described above, it appears in practice that it is very rapid if the two guessed values are in opposite sides of the true value. Otherwise also, when the chosen value is nearer to the true value the process converges rapidly).

The algorithm may be applied to almost all two-point boundary value problems whose order is higher than three provided that only one condition from the minimum number of required initial conditions at one end point is missing.

## 4 Application of Algorithm to (3) to (4)

Since it is not easy to form an analytical method to look for the guess values, in our present problem we consider a particular value of $\beta$ and try to get two guess values say $\lambda_0$ and $\lambda_1$ for the missing initial condition $f''(0)$ corresponding to that value of $\beta$. In this selection a trial and error method is used. It is experienced that once the inaration method is convergent with above $\lambda_0$ and $\lambda_1$ for that value to $\beta$, the selection of the guessed values corresponding to any other value of $\beta$ becomes much easier.

Since in (3) to (4) one of the boundary is $\eta=\infty$, the problem becomes a singular one. The boundary $\eta=\infty$ is tackled in the following manner. First for the fixed $\beta$, a large value $\eta= \eta_\infty$ (say) is chosen and corresponding initial condition $f''(0)$ is determined. Let this estimated value of $f''(0)$ be $\lambda_i$. We go on increasing the value of $\eta_\infty$ after estimating $f''(0)$ at each value until a fixed value of $f''(0)$ is attained. Then say $\eta_\infty$ for which $f''(0)$ takes that fixed value is considered as the truncated boundary for $\eta=\infty$. If the guessed value to $\eta_\infty$ is large enough ( say 5 here) the process for determining the boundary $\eta=\infty$ is not repeated more than two times.

## 5 Estimation of the Parameter B by Shooting Method

It has been possible to estimate the parameter by Shooting Method. Knowing all the initial conditions the parameter $\beta$ can be estimated in such as way that the resulting estimation satisfy $f'(\infty)=1$ to some desired accuracy in a similar manner to that of estimation of $f''(0)$.

Since $f'(\infty)$ is a function of $\beta$ a similar geometry can be constructed by plotting $f'(\infty)$ against $\beta$. If $\beta_0$ and $\beta_1$ are approximate values of $\beta$ for $f(0) = 0$ , $f'(0) =0$ and for fixed $f''(0) = \infty$ (say) then the corresponding iteration formula for the next approximation $\beta_2$ of $\beta$ are given by

$$\beta_2 = \beta_0 + (\beta_1 - \beta_0) \{ 1 - f'( \infty , \beta_0 )\} / \{ f' ( \infty , \beta_1 ) - f' ( \infty , \beta_0 )\} \quad (6)$$

where we put $f'(\infty) = 1$ and $f' (\infty,\beta_k)$, (k= 0,1) are the values of $f'(\infty)$ obtained on integration the initial value problem with $f''(0) = \infty$ (known) and $\beta = \beta_k$ . Then using $\beta = \beta_2$ , the equation is again integrated to obtain $f'(\infty, \beta_2)$ and linear iteration based on $\beta_1$ and $\beta_2$ a better approximation $\beta_3$ can be obtained. The process is repeated until the resulting approximation of $\beta$ satisfied $f'(\infty) = 1$ to some desired places.

Here also the convergence of the iteration process is not guaranteed but once the chosen value is nearer to the true value the process converges very rapidly. Also if the two guessed values are in opposite sides of the true value the convergence is very rapid.

Accordingly we have presented below an algorithm for estimation of $\beta$.

### 5.1 Algorithm

Step 1 : let $\beta_0$ and $\beta_1$ be two approximate values of for some known $f''(0) = \infty$ (say).

Step 2: integrate the initial value problem
$f''' + f f'' + \beta_k (1- f'^2) = 0$
with $f(0) = 0$ , $f'(0) =0$ , $f''(0) = \infty$, k= 0, 1, ....
from $\eta = 0$ to $\eta = \infty$ ( the boundary $\eta = \infty$ is now easy to determine for known $f''(0)$.

Step 3: Call $f'(\infty, \beta_k)$ the value of $f'(\infty)$ obtained from step 2 on integration with $\beta=\beta_k$,

Step 4 : if $|1- f'(\infty,\beta_k)| < \varepsilon$ for some prescribed $\varepsilon$, then go to step 6 otherwise continue.

Step 5: Obtain next approximation from
$\beta_{k+2} = \beta_k + (\beta_{k+1} - \beta_k) \{1 - f'(\infty, \beta_k)\} / \{f'(\infty, \beta_{k+1}) - f'(\infty, \beta_k)\}$ and go to step 2.

Step 6: Solve (2.1) as initial value problem with $f(0) = 0$, $f'(0) = 0$, $f''(0) = \infty$ , $\beta=\beta_k$ and stop.

We have tested the algorithm for some known $f''(0)$ and known $\beta$. Then it becomes easier to guess the two values of $\beta$ for any other values of $f''(0)$. The boundary $\eta=\infty$ is now very easy to tackle.

## 6 Lower Bound of $\beta$

It is proved analytically that the upper bound of does not exist. For $\beta< 0$, $f''(0)$ satisfies the inequality (2.3). For problem (2.1) to (2.2), the number a defined in section 2.3 depends only on $\beta$, and (2.3) takes the form $0 \le f''(0) \le h(0)$ since a = 0 here. Therefore the critical value of $\beta$ is related to the condition $f''(0) = 0$. There are two ways to determine the least value of $\beta$ for which solution of (2.1) and (2.2) exists. Firstly the number A should be chosen in such as way that $A(\beta) = 0$.

Then solving this equation for $\beta$ one can obtain the least value of $\beta$. Secondly, if $A(\beta) = 0$ holds, then by definition of h, $h(A) = h(a) = h(0) = 0$ and hence $f''(0) =0$ for the least value of $\beta$.

Hence the critical value of $\beta$ is such that, for that particular value of $\beta$, $f''(0) = 0$ holds. So one can estimate $\beta$ for which $f''(0) = 0$.

It is not easy to get an analytical expression for $A(\beta)$ , which satisfies $A(\beta) = 0$, so that the critical value of $\beta$ can be determined. Also we are in a position to estimate the parameter $\beta$ for known $f''(0)$, so we take the second approach, to determine the least value of $\beta$. Using the second algorithm for $f''(0) =0$, we obtain $\beta = - 0.19884$ which is the critical value of the parameter $\beta$.

## 7 Results and Discussions

The two guessed values needed for the algorithms to estimate $f''(0)$ and $\beta$ are chosen by trial and error methods. Once the correct values are obtained for some cases e.g. for some prescribed $\beta$ in estimation of $f''(0)$ and for some prescribed $f''(0)$ in estimation of $\beta$ then it becomes easier to choose the guess values for any other values of $\beta$ or $f''(0)$ as the case may be.

The values if f, $f'$ and $f''$ for $\beta$ = 0, ½, 1 obtained by shooting method agree up to three places of decimal with those of Blasius [3], Frossling [4] and Hiemenz [5]. For $\beta$ = -0.1988, values of f, $f'$ and $f''$ are presented in table.

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**Appendix A**

**β=-0.1988**

| η | f | f' | f'' |
|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.008416 |
| 0.2 | 0.000433 | 0.005659 | 0.048175 |
| 0.4 | 0.002794 | 0.019268 | 0.087908 |
| 0.6 | 0.008670 | 0.040814 | 0.127512 |
| 0.8 | 0.019646 | 0.070249 | 0.166740 |
| 1.0 | 0.037288 | 0.107456 | 0.205139 |
| 1.2 | 0.063131 | 0.152203 | 0.241999 |
| 1.4 | 0.098646 | 0.204087 | 0.276324 |
| 1.6 | 0.145201 | 0.262479 | 0.306832 |
| 1.8 | 0.204011 | 0.326465 | 0.332012 |
| 2.0 | 0.276079 | 0.394820 | 0.350241 |
| 2.2 | 0.362128 | 0.465995 | 0.359983 |
| 2.4 | 0.462544 | 0.538165 | 0.360032 |
| 2.6 | 0.577326 | 0.609319 | 0.349788 |
| 2.8 | 0.706066 | 0.677407 | 0.329487 |
| 3.0 | 0.847956 | 0.740520 | 0.300318 |
| 3.2 | 1.001840 | 0.797079 | 0.264364 |
| 3.4 | 1.166280 | 0.845994 | 0.224361 |
| 3.6 | 1.339969 | 0.886753 | 0.183301 |
| 3.8 | 1.520440 | 0.919431 | 0.143987 |
| 4.0 | 1.706960 | 0.944613 | 0.108648 |
| 4.2 | 1.089785 | 0.963249 | 0.078699 |
| 4.4 | 2.091900 | 0.976488 | 0.054699 |
| 4.6 | 2.288816 | 0.985511 | 0.036472 |
| 4.8 | 2.485900 | 0.991411 | 0.023327 |
| 5.0 | 2.684590 | 0.995113 | 0.014311 |
| 5.2 | 2.883850 | 0.997340 | 0.008421 |
| 5.4 | 3.083460 | 0.998625 | 0.004753 |
| 5.6 | 3.283260 | 0.999337 | 0.002573 |
| 5.8 | 3.483170 | 0.999715 | 0.001335 |
| 6.0 | 3.683130 | 0.999907 | 0.000663 |
| 6.2 | 3.883120 | 1.000000 | 0.000314 |